Totally Gauge-Invariant Formulation of Perturbed FRW Cosmologies (and Hybrid Quantization)

Mercedes Martín-Benito

Radboud University Nijmegen

[L. Castelló Gomar, M. M-B, G.A. Mena Marugán, JCAP 06, 045, 2015]

8th Jerte Advanced Relativity Workshop
[24/05/2016]

INTRODUCTION

- Our Universe is approximately homogeneous and isotropic:
  $\longrightarrow$ FLRW background with cosmological **perturbations**

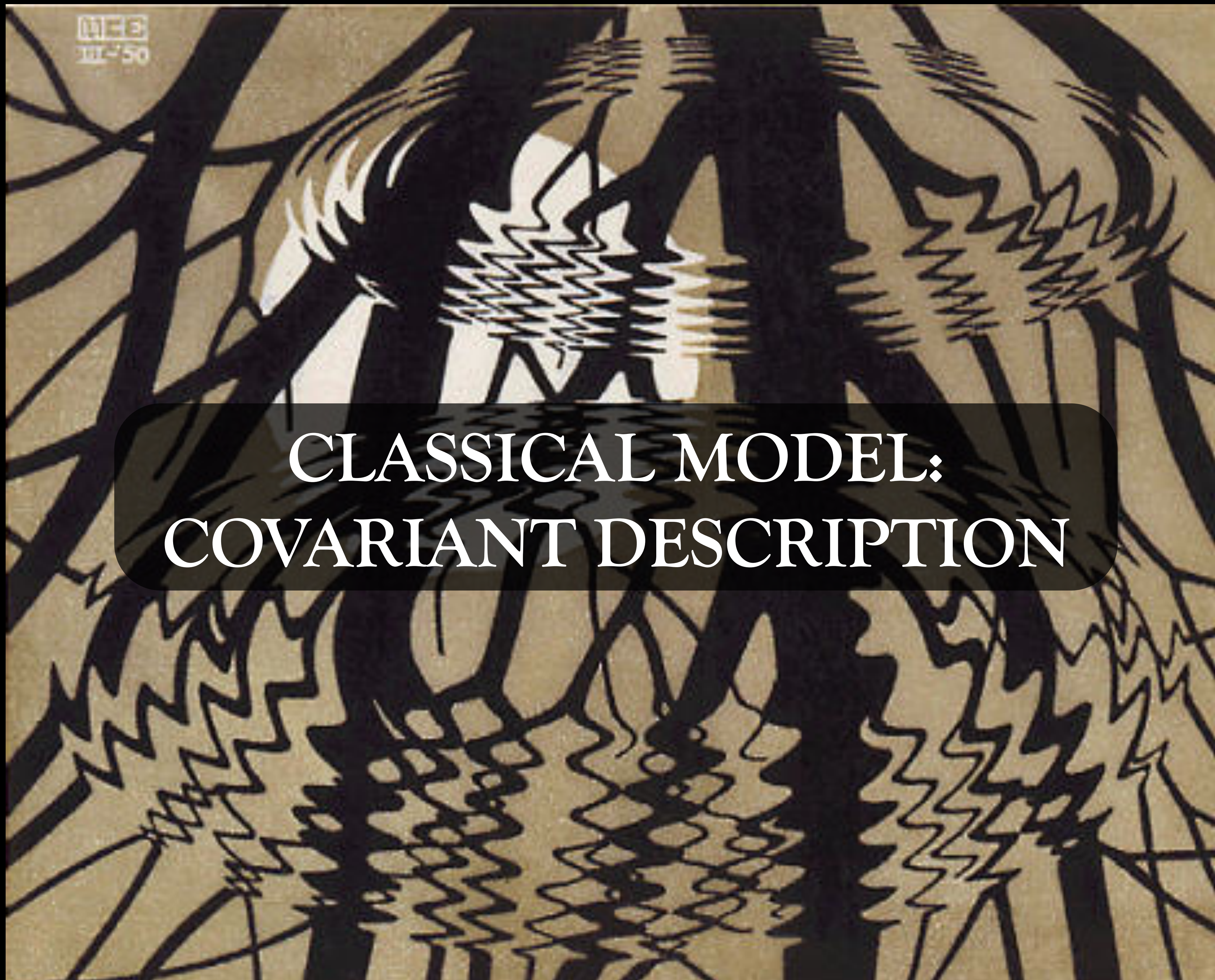
- Need of **gauge-invariant** description [Bardeen, Sasaki, Mukhanov,...]
  [Only for perturbations, treated as a test field in fixed background]

- **Canonical formulation** with constraints [Langlois, Pinto-Neto,...]
  [Fixed background, or appearance of second class constraints]

- **Quantization** including the background [Halliwell-Hawking, Shirai-Wada...]
  [No totally gauge-invariant description]

- **Hybrid** quantization in the framework of **LQC**
  [Gauge-fixing] [Fernández Méndez, Mena Marugán, Olmedo, Castelló Gomar]

- Here: **Hybrid quantization preserving covariance**



CLASSICAL MODEL: COVARIANT DESCRIPTION

CLASSICAL SYSTEM

- We consider a **FLRW** model with flat **compact topology**

- We include a **scalar field** subject to a potential (e.g. mass term)

- For simplicity, we analyze only **scalar perturbations**

Mode decomposition

- We expand the inhomogeneities in (real) **Fourier modes**

$$Q_{\vec{n},+}(\vec{\theta}) = \sqrt{2}\cos(\vec{n}\cdot\vec{\theta}) \qquad Q_{\vec{n},-}(\vec{\theta}) = \sqrt{2}\sin(\vec{n}\cdot\vec{\theta}) \qquad \vec{n}\in\mathbb{Z}^3$$

- We take the first entry in $\vec{n}$ to be non-negative

- The eigenvalue of the Laplacian is $-\omega_n^2 = -\vec{n}\cdot\vec{n}$

METRIC AND FIELD

[Halliwell, Hawking]

- Expansion up to **linear order** in perturbations

- **Zero-modes** are treated exactly (at linear order) $\quad (\vec{n} \neq (0,0,0))$

$$h_{ij} = \sigma^2\, e^{2\alpha} \Big[{}^0h_{ij} + 2 \sum a_{\vec{n},\pm}(t) Q_{\vec{n},\pm}\, {}^0h_{ij} \\ + 2 \sum b_{\vec{n},\pm}(t) \Big(\frac{3}{\omega_n^2} (Q_{\vec{n},\pm})_{;ij} + Q_{\vec{n},\pm}\, {}^0h_{ij} \Big) \Big],$$

$$N = \sigma \Big[N_0(t) + e^{3\alpha} \sum g_{\vec{n},\pm}(t) Q_{\vec{n},\pm} \Big], \quad N_i = \sigma^2\, e^{2\alpha} \sum k_{\vec{n},\pm}(t) \frac{1}{\omega_n^2} (Q_{\vec{n},\pm})_{;i},$$

$$\Phi = \frac{1}{\sigma (2\pi)^{3/2}} \Big[\varphi(t) + \sum f_{\vec{n},\pm}(t) Q_{\vec{n},\pm} \Big], \qquad \sigma^2 \propto G$$

HAMILTONIAN

- Truncating at **quadratic perturbative order** in the action

$$H = N_0 \left[H_0 + \sum H_2^{\vec{n},\pm} \right] + \sum g_{\vec{n},\pm} \, H_{|1}^{\vec{n},\pm} + \sum k_{\vec{n},\pm} \, H_{-1}^{\vec{n},\pm}$$

Quadratic in perturbations

Linear perturbative constraints: Come from perturbing the scalar and diffeos constraints

- The zero-mode of the Hamiltonian constraint gets corrections quadratic in perturbations

- Symplectic structure for the entire system: zero-modes + perturbations

GAUGE-INVARIANT PERTURBATIONS

- Consider the zero-modes as describing a fixed **background**

- Transformation of the perturbations, canonical only w.r.t. their symplectic structure, adapted to gauge-invariance

a) Abelianization of perturbative constraints

$$H^{\vec{n},\pm}_{|1} \to \breve{H}^{\vec{n},\pm}_{|1} = H^{\vec{n},\pm}_{|1} - 3e^{3\alpha} H_0 a_{\vec{n},\pm} \quad , \quad \{H^{\vec{n},\pm}_{-1}, \breve{H}^{\vec{n},\pm}_{|1}\} = 0$$

GAUGE-INVARIANT PERTURBATIONS

- Consider the zero-modes as describing a fixed **background**

- Transformation of the perturbations, canonical only w.r.t. their symplectic structure, adapted to gauge-invariance

a) Abelianization of perturbative constraints , $\{H_{-1}^{\vec{n},\pm}, \breve{H}_{|1}^{\vec{n},\pm}\} = 0$

b) Include Mukhanov-Sasaki gauge invariant , $v_{\vec{n},\pm}$

$$v_{\vec{n},\pm} = e^{\alpha}[f_{\vec{n},\pm} + (a_{\vec{n},\pm} + b_{\vec{n},\pm})\pi_{\varphi}/\pi_{\alpha}] \quad , \quad \{v_{\vec{n},\pm}, \breve{H}_{|1}^{\vec{n},\pm}\} = \{v_{\vec{n},\pm}, H_{-1}^{\vec{n},\pm}\} = 0$$

GAUGE-INVARIANT PERTURBATIONS

- Consider the zero-modes as describing a fixed **background**

- Transformation of the perturbations, canonical only w.r.t. their symplectic structure, adapted to gauge-invariance

a) Abelianization of perturbative constraints , $\{H_{-1}^{\vec{n},\pm}, \breve{H}_{|1}^{\vec{n},\pm}\} = 0$

b) Include Mukhanov-Sasaki gauge invariant , $v_{\vec{n},\pm}$

c) Complete this canonical transformation

$$\{X_{q_l}^{\vec{n},\pm}\} = \{a_{\vec{n},\pm}, b_{\vec{n},\pm}, f_{\vec{n},\pm}\} \to \{V_{q_l}^{\vec{n},\pm}\} = \{v_{\vec{n},\pm}, C_{-1}^{\vec{n},\pm}, \breve{C}_{|1}^{\vec{n},\pm}\}$$

$$\{X_{p_l}^{\vec{n},\pm}\} = \{\pi_{a_{\vec{n},\pm}}, \pi_{b_{\vec{n},\pm}}, \pi_{f_{\vec{n},\pm}}\} \to \{V_{p_l}^{\vec{n},\pm}\} = \{\pi_{v_{\vec{n},\pm}}, H_{-1}^{\vec{n},\pm}, \breve{H}_{|1}^{\vec{n},\pm}\}$$

GAUGE-INVARIANT PERTURBATIONS

- The redefinition of the perturbative scalar constraint amounts to a **redefinition of the lapse** at our order of truncation

$$H_{|1}^{\vec{n},\pm} \to \breve{H}_{|1}^{\vec{n},\pm} = H_{|1}^{\vec{n},\pm} - 3e^{3\alpha} H_0 a_{\vec{n},\pm}$$

$$N_0 \to \breve{N}_0 = N_0 + 3e^{3\alpha} \sum g_{\vec{n},\pm} a_{\vec{n},\pm}$$

$$H = \breve{N}_0 \left[H_0 + \sum H_2^{\vec{n},\pm} \right] + \sum g_{\vec{n},\pm} \breve{H}_{|1}^{\vec{n},\pm} + \sum k_{\vec{n},\pm} H_{-1}^{\vec{n},\pm}$$

FULL SYSTEM

- We now include the **zero modes** as variables of the system, and complete the transformation to a **canonical** one

- We rewrite the Legendre term of the action, keeping its canonical form at the considered **perturbative order**

$$\int dt\Big[\sum_a \dot{w}_q^a w_p^a + \sum_{l,\vec{n},\pm} \dot{X}_{q_l}^{\vec{n},\pm} X_{p_l}^{\vec{n},\pm}\Big] \equiv \int dt\Big[\sum_a \dot{\tilde{w}}_q^a \tilde{w}_p^a + \sum_{l,\vec{n},\pm} \dot{V}_{q_l}^{\vec{n},\pm} V_{p_l}^{\vec{n},\pm}\Big]$$

- Zero-modes: Old $\{w_q^a, w_p^a\} \longrightarrow$ New $\{\tilde{w}_q^a, \tilde{w}_p^a\}$ $\Big(w_q^a = (\alpha, \varphi)\Big)$

- Inhomogeneities: Old $\{X_{q_l}^{\vec{n},\pm}, X_{p_l}^{\vec{n},\pm}\} \longrightarrow$ New $\{V_{q_l}^{\vec{n},\pm}, V_{p_l}^{\vec{n},\pm}\}$

$$\{V_{q_l}^{\vec{n},\pm}, V_{p_l}^{\vec{n},\pm}\} = \{(v_{\vec{n},\pm}, C_{-1}^{\vec{n},\pm}, \breve{C}_{|1}^{\vec{n},\pm}), (\pi_{v_{\vec{n},\pm}}, H_{-1}^{\vec{n},\pm}, \breve{H}_{|1}^{\vec{n},\pm})\}$$

NEW HAMILTONIAN

- The difference between old and new **zero-modes** is **quadratic in the perturbations**

- The new **scalar constraint** at our truncation order is

$$H_0(w^a) + \sum_{\vec{n},\epsilon} H_2^{\vec{n},\epsilon}\left(w^a, X_l^{\vec{n},\epsilon}\right) \longrightarrow H_0(\tilde{w}^a) + \sum_{\vec{n},\epsilon} H_2^{\vec{n},\epsilon}\left[\tilde{w}^a, X_l^{\vec{n},\epsilon}(\tilde{w}^a, V_l^{\vec{n},\epsilon})\right] + \sum_b \left(w^b - \tilde{w}^b\right)\frac{\partial H_0}{\partial \tilde{w}^b}(\tilde{w}^a)$$

- The new **Hamiltonian** at our truncation order is

$$H = \bar{N}_0\left[H_0 + \sum \breve{H}_2^{\vec{n},\pm}\right] + \sum G_{\vec{n},\pm}\breve{H}_{|1}^{\vec{n},\pm} + \sum K_{\vec{n},\pm}H_{-1}^{\vec{n},\pm}$$

Redefined Lagrange multipliers

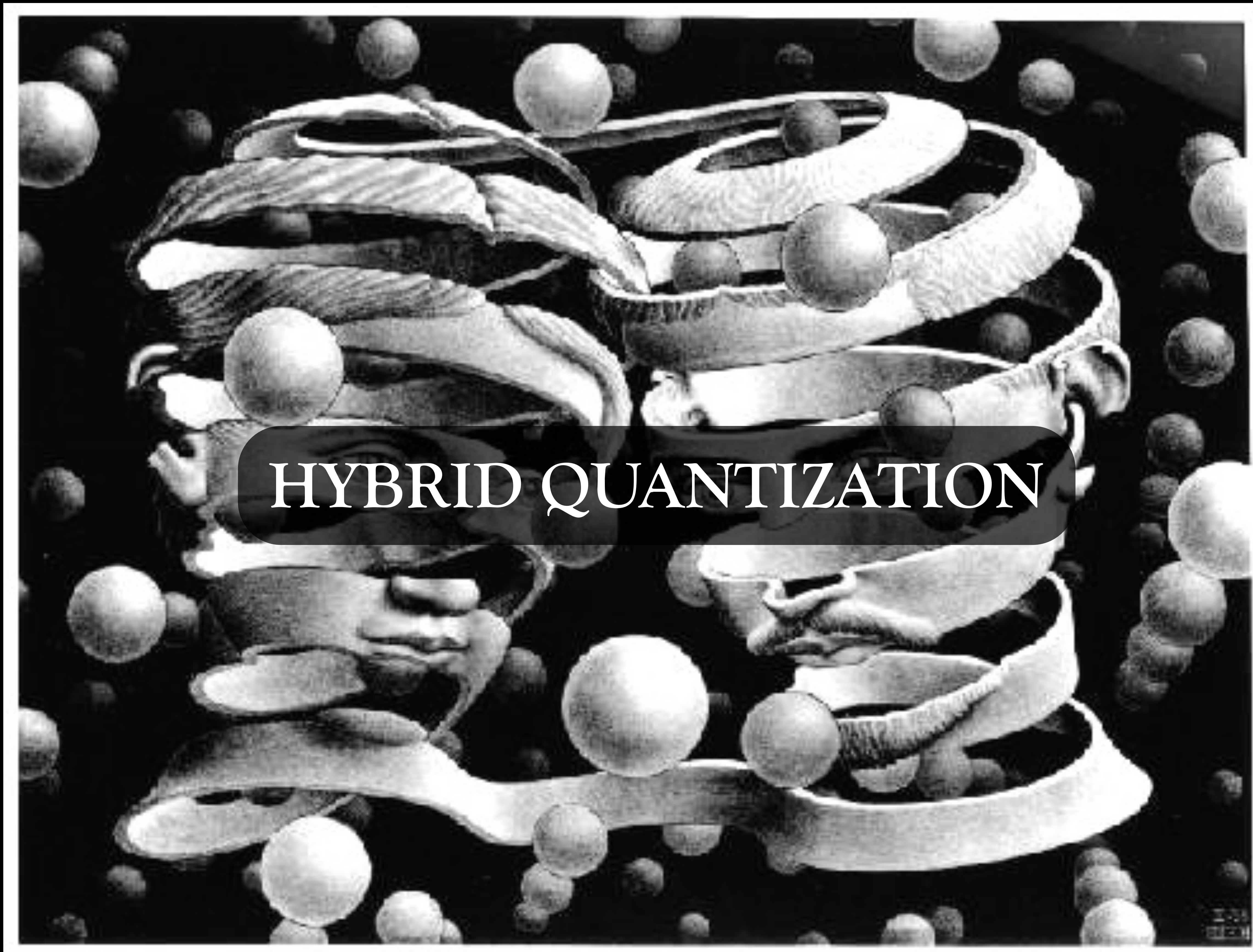
NEW HAMILTONIAN

$$H = \bar{N}_0 \left[H_0 + \sum \breve{H}_2^{\vec{n},\pm} \right] + \sum G_{\vec{n},\pm} \breve{H}_{|1}^{\vec{n},\pm} + \sum K_{\vec{n},\pm} H_{-1}^{\vec{n},\pm}$$

- The term $\breve{H}_2^{\vec{n},\pm}$ is the Mukhanov-Sasaki Hamiltonian

- It has quadratic contributions both in the Mukhanov-Sasaki field and in its momentum

- It accounts for the **backreaction** at the considered perturbative order



HYBRID QUANTIZATION

HYBRID QUANTIZATION

[Garay, Martín-Benito, Mena Marugán]

- **Fock** representation for the **inhomogeneous** sector

- **QG** inspired rep. for the **homogeneous geometry** sector: special treatment for the zero-mode of the conformal factor

Assumption:

- Main quantum gravity effects are those affecting the global degrees of freedom.

- Inhomogeneities, even though are also quantum, can be treated in a more conventional way.

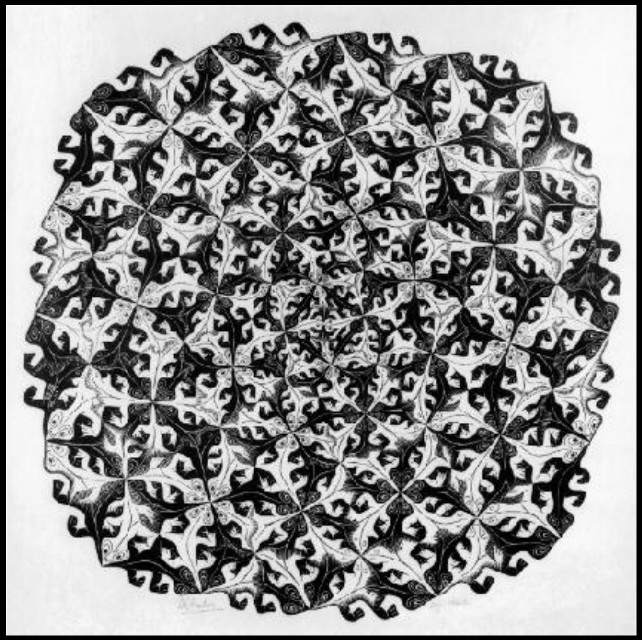
By quantizing the homogeneous sector with quantum gravity techniques, we hope to retain the main quantum geometry effects

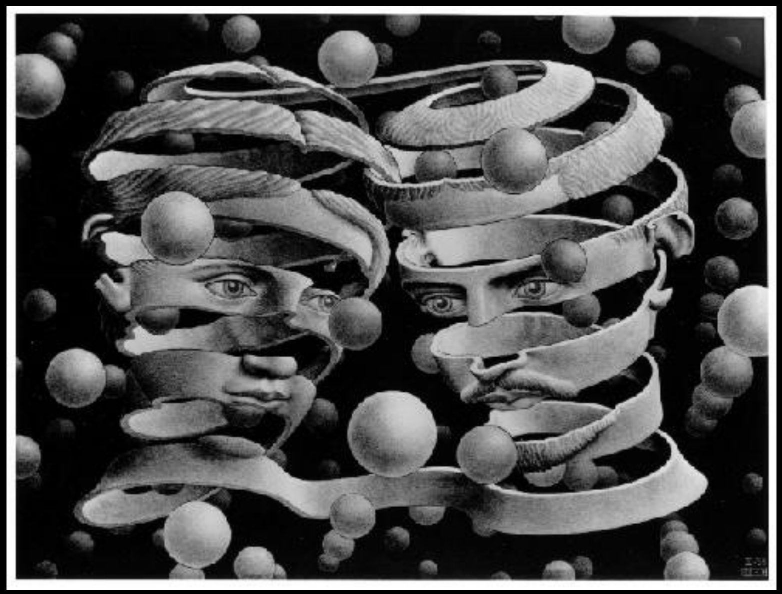
HYBRID QUANTIZATION

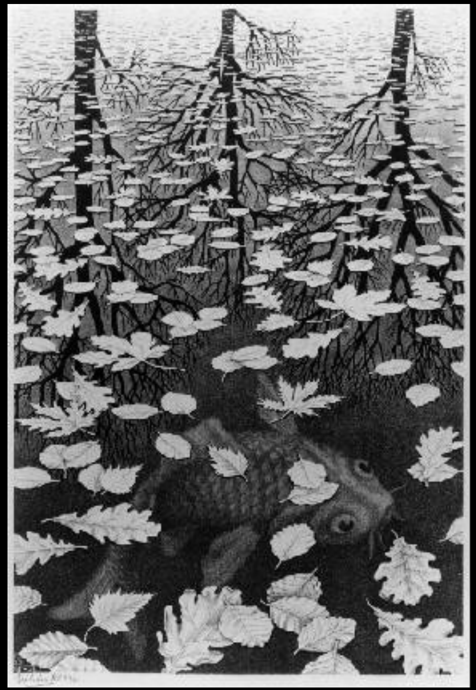
[Garay, Martín-Benito, Mena Marugán]

If we believe in QFT on curved spacetimes, it makes sense to think that there is a deeper quantum regime, before the full quantum gravity regime, where our hybrid approach is valid

QG ⟶ **Hybrid Q** ⟶ **QFT/CS**







INHOMOGENEITIES: FOCK QUANTIZATION

- When dealing with the Fock quantization of free (test) fields, propagating in homogeneous spacetimes with compact spatial sections, the criteria of **invariance** of the vacuum under the **spatial isometries** and of a **unitary** implementation of the quantum **dynamics** select a **unique equivalence class** of Fock representations.

- Parameterization for the M-S field selected by those results, which involves a particular time-dependent scaling.

- Within the family of unitarily equivalent Fock reps. selected, we choose e.g. that corresponding to the massless scalar field.

HOMOGENEOUS SECTOR

- Geometry: $\{b, v\} = 2$, $V \propto |v|$, $H \propto b$

- Matter: $\{\tilde{\varphi}, \pi_{\tilde{\varphi}}\} = 1$, Potential $W(\tilde{\varphi})$

- Hilbert space: $\mathcal{H}_{\text{hom}} = \mathcal{H}_v \otimes L^2(\mathbb{R}, d\tilde{\varphi})$

- Hamiltonian constraint:

$$\boxed{\hat{H}_0 = \frac{\hat{\pi}_{\tilde{\varphi}}^2}{2} - \frac{9}{32}\hat{\Omega}^2 + \left[\frac{3}{4\pi G}\hat{V}\right]^2 W(\hat{\tilde{\varphi}}) \equiv \frac{1}{2}\left(\hat{\pi}_{\tilde{\varphi}}^2 - \hat{h}_0^2\right)}$$

$\hat{\Omega}^2$ **represents** $(2vb)^2$

QUANTUM CONSTRAINTS

- We represent the **linear perturbative constraints** (or an integrated version of them) as derivatives (or as translations)

- Then, physical states are independent of $\left(C_{-1}^{\vec{n},\pm}, \breve{C}_{|1}^{\vec{n},\pm}\right)$

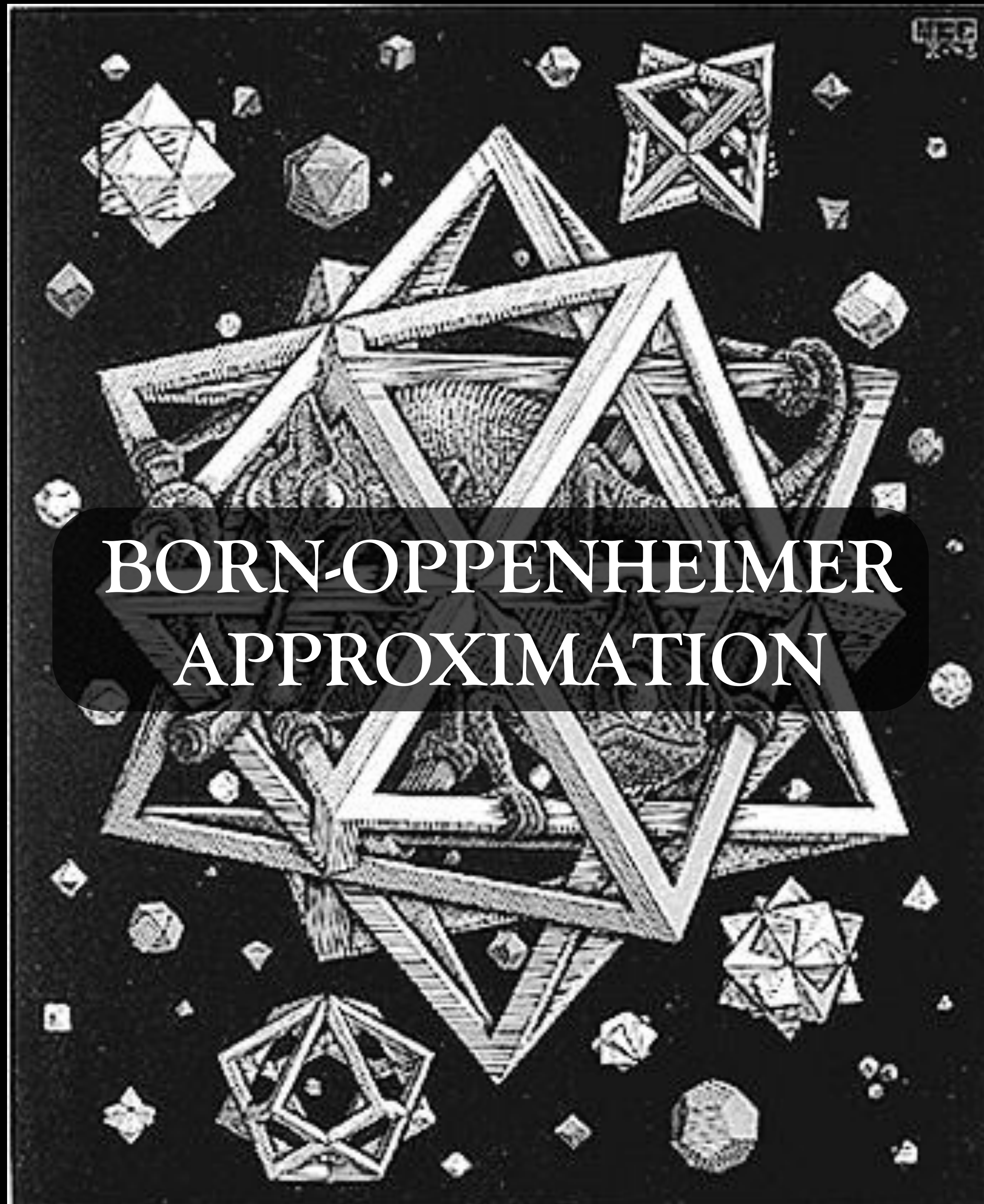
- We pass to a space of states $\mathcal{H}_{\text{hom}} \otimes \mathcal{F}$, that depend on the **zero-modes** and the **Mukhanov-Sasaki modes**, with **no gauge fixing**

- Physical states still must satisfy the scalar constraint

$$\hat{H}_{\text{S}} = \frac{1}{2}\left[\hat{\pi}_{\tilde{\varphi}}^2 - \hat{h}_0^2 - \hat{\Theta}_e - \frac{1}{2}\left(\hat{\Theta}_o \hat{\pi}_{\tilde{\varphi}} + \hat{\pi}_{\tilde{\varphi}} \hat{\Theta}_o\right)\right]$$

Quadratic in M-S modes and do not depend on the field momentum

BORN-OPPENHEIMER APPROXIMATION



BORN-OPPENHEIMER ANSATZ

- Consider states whose dependance on the FLRW geometry and the inhomogeneities ($\mathcal{N}$) **split**

$$\Psi = \Gamma(v,\tilde{\varphi})\psi(\mathcal{N},\tilde{\varphi})$$

- The FLRW state is **normalized**, **peaked**, and **evolves unitarily**

$$\Gamma(v,\tilde{\varphi}) = \hat{U}(v,\tilde{\varphi})\chi(\tilde{\alpha})$$

$\tilde{\varphi}$ internal time

$$\hat{U}(v;\tilde{\varphi}) = \mathcal{P}\left[\exp\left(i\int_{\tilde{\varphi}_0}^{\tilde{\varphi}} d\tilde{\varphi}\,\hat{h}_0(v,\tilde{\varphi})\right)\right]$$

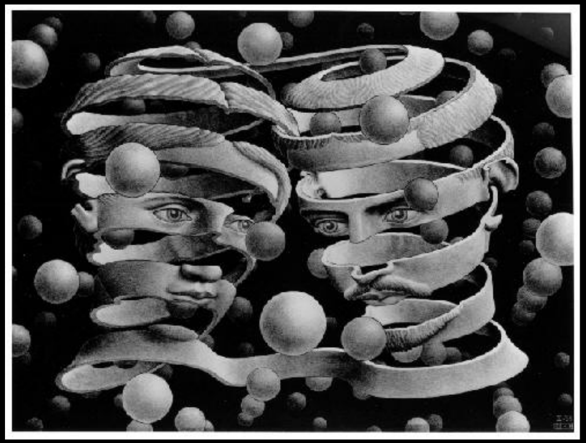
EVOLUTION FOR PERTURBATIONS

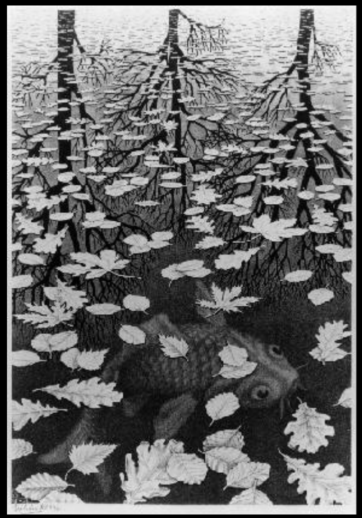
- Profiles such that we can disregard: i) transitions from Γ to other FLRW states, ii) $\hat{\pi}^2_{\tilde{\varphi}}\psi$, iii) $[\hat{\pi}_{\tilde{\varphi}} - \hat{h}_0, \hat{\Theta}_o]$

$$\boxed{-i\partial_{\tilde{\varphi}}\psi = \frac{\langle \hat{\Theta}_e + \big(\hat{\Theta}_o \hat{h}_0 + \hat{h}_0 \hat{\Theta}_o\big)/2 \rangle_\Gamma}{2\langle \hat{h}_0 \rangle_\Gamma}\psi}$$

Schrödinger Eq. for the gauge-invariant perturbations in a fixed background with quantum gravity corrections

Hybrid Q $\longrightarrow$ **QFT/CS**





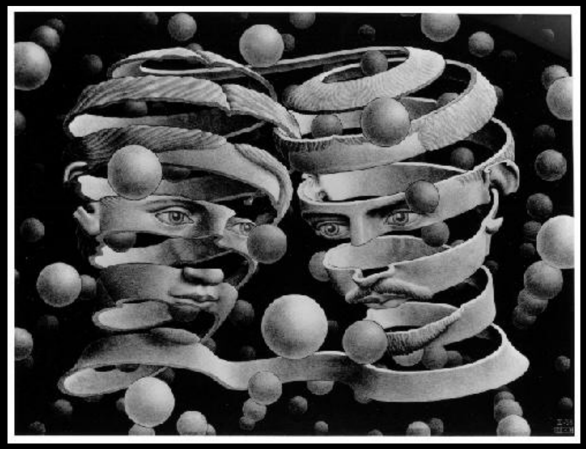
EVOLUTION FOR PERTURBATIONS

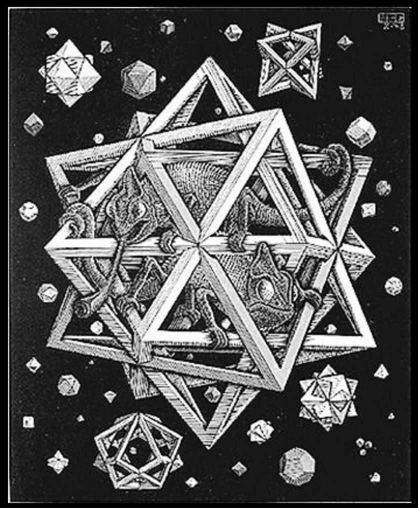
- Profiles such that we can disregard: i) transitions from Γ to other FLRW states, ii) $\hat{\pi}^2_{\tilde{\varphi}}\psi$, iii) $[\hat{\pi}_{\tilde{\varphi}} - \hat{h}_0, \hat{\Theta}_o]$

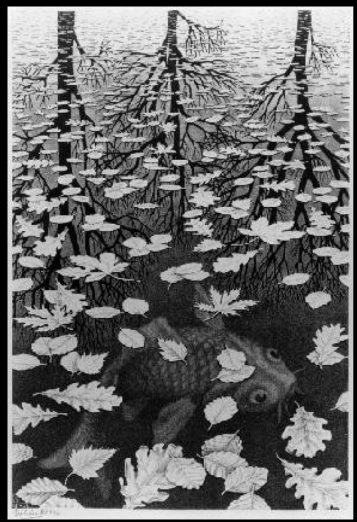
$$\boxed{-i\partial_{\tilde{\varphi}}\psi = \frac{\langle\hat{\Theta}_e + \big(\hat{\Theta}_o\hat{h}_0 + \hat{h}_0\hat{\Theta}_o\big)/2\rangle_\Gamma}{2\langle\hat{h}_0\rangle_\Gamma}\psi}$$

Schrödinger Eq. for the gauge-invariant perturbations in a fixed background with quantum gravity corrections

Hybrid Q $\longrightarrow$ **QFT/quantum CS** $\longrightarrow$ **QFT/CS**







EFFECTIVE M-S EQUATIONS

- Considering the Born-Oppenheimer ansatz, that assumption i) holds, and the perturbations as classical:

$$\boxed{d^2_{\eta_\Gamma} v_{\vec{n},\epsilon} = -v_{\vec{n},\epsilon} 4\pi^2 \left[\omega_n^2 + \frac{\langle \hat{\vartheta}_e^q + \frac{1}{2}(\hat{\vartheta}_o \hat{h}_0 + \hat{h}_0 \hat{\vartheta}_o) + \frac{1}{2}[\hat{\pi}_{\tilde{\varphi}} - \hat{h}_0 \,, \hat{\vartheta}_o] \rangle_\Gamma}{\langle \hat{\vartheta}_e \rangle_\Gamma} \right]}$$

state-dependent conformal time $\quad 2\pi \, d\eta_\Gamma = \langle \hat{\vartheta}_e \rangle_\Gamma dT \quad \left(\frac{d}{dT} = \{ \quad , H_S \} \right)$

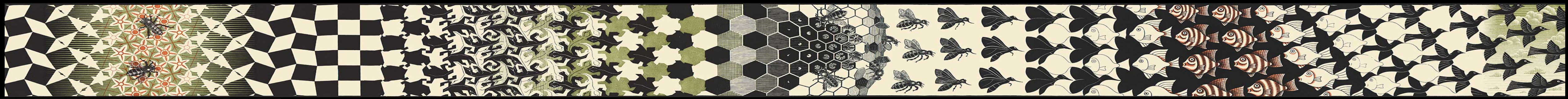
- The effective equations are of **harmonic oscillator** type, with time-dependent frequency, no dissipative term, and **hyperbolic in the ultraviolet** regime

- **Master Eq. to predict QG modifications to observables**

CONCLUSIONS

Challenge for quantum cosmology:

To build a formalism which includes the homogeneous background and the inhomogeneities, and proves to be potentially predictive.



CONCLUSIONS

- We have considered a FLRW universe with a scalar field perturbed at **quadratic** order in the action

- We have found a canonical transformation for the **full system** that respects **covariance** at the perturbative level of truncation

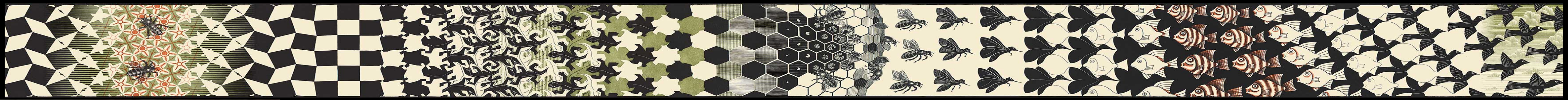
- **Constrained symplectic manifold**, where zero-modes incorporate quadratic contributions of the perturbations

- **Backreaction** is included at the considered perturbative order in the scalar constraint, by means of the **M-S Hamiltonian**

M.C. ESCHER

CONCLUSIONS

- We have discussed the **hybrid quantization** of the system
- Physical states depend only on the **zero-modes** and the **Mukhanov-Sasaki field**
- A **Born-Oppenheimer** ansatz leads to a quantum evolution equation for the inhomogeneities, without recurring to **any semiclassical approximation**
- We have derived effective **Mukhanov-Sasaki equations**, which include quantum corrections
- Master equations to extract physical consequences of quantum gravity in cosmology



M.C. ESCHER

THANK YOU FOR YOUR ATTENTION!

M.C. ESCHER